Chebyshev

Theorem 1.1 *(Chebyshev) Suppose X is a random variable with finite expectation, μ, and variance, σ^2. Then for any real $\epsilon > 0$*

$$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \tag{1}$$

or

$$P(|X - \mu| < \epsilon) \geq 1 - \frac{\sigma^2}{\epsilon^2} \tag{2}$$

The following figure illustrates this point.

Figure 1: Chebyshev

Proof 1.1 *By definition*

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx \tag{3}$$

Following Figure 1, we have

$$\sigma^2 = \int_{-\infty}^{\mu-\epsilon} (x-\mu)^2 f_X(x)dx + \int_{\mu-\epsilon}^{\mu+\epsilon} (x-\mu)^2 f_X(x)dx + \int_{\mu+\epsilon}^{\infty} (x-\mu)^2 f_X(x)dx \tag{4}$$

Note that

$$\int_{\mu-\epsilon}^{\mu+\epsilon} (x-\mu)^2 f_X(x)dx \geq 0 \tag{5}$$

so that

$$\sigma^2 \geq \int_{-\infty}^{\mu-\epsilon} (x-\mu)^2 f_X(x)dx + \int_{\mu+\epsilon}^{\infty} (x-\mu)^2 f_X(x)dx \tag{6}$$

As x tends from $-\infty$ to $\mu-\epsilon$ and from $\mu+\epsilon$ to ∞ the smallest values of $(x-\mu)^2$ are attained for $\mu \pm \epsilon$. Hence,

$$\begin{aligned}
\sigma^2 &\geq \int_{-\infty}^{\mu-\epsilon} \epsilon^2 f_X(x)dx + \int_{\mu+\epsilon}^{\infty} \epsilon^2 f_X(x)dx && (7)\\
&= \epsilon^2 \left[\int_{-\infty}^{\mu-\epsilon} f_X(x)dx + \int_{\mu+\epsilon}^{\infty} f_X(x)dx\right] && (8)\\
&= \epsilon^2 \left[P(X \leq \mu-\epsilon) + P(X \geq \mu+\epsilon)\right] && (9)\\
&= \epsilon^2 P(|X-\mu| \geq \epsilon) && (10)
\end{aligned}$$

Thus

$$P(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \tag{11}$$

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Alternatively, letting $\epsilon = h\sigma$ for $h > 0$ we have

$$\mathrm{P}(|X-\mu| \geq h\sigma) \leq \frac{1}{h^2} \tag{12}$$

Choosing a different h allows us to find upper bounds on the probability that X takes on a value outside the interval $\mu \pm h\sigma$. We get that, irrespective of the distribution of X, this probability is at most $\frac{1}{h^2}$.